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Optimal investment models with vintage capital: Dynamic Programming approach

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Abstract

The Dynamic Programming approach for a family of optimal investment models with vintage capital is here developed. The problem falls into the class of infinite horizon optimal control problems of PDE's with age structure that have been studied in various papers (see e.g. [11, 12], [30, 32]) either in cases when explicit solutions can be found or using Maximum Principle techniques.

The problem is rephrased into an infinite dimensional setting, it is proven that the value function is the unique regular solution of the associated stationary Hamilton–Jacobi–Bellman equation, and existence and uniqueness of optimal feedback controls is derived. It is then shown that the optimal path is the solution to the closed loop equation. Similar results were proven in the case of finite horizon in [26][27]. The case of infinite horizon is more challenging as a mathematical problem, and indeed more interesting from the point of view of optimal investment models with vintage capital, where what mainly matters is the behavior of optimal trajectories and controls in the long run.

The study of infinite horizon is performed through a nontrivial limiting procedure from the corresponding finite horizon problems.

Keywords. Optimal investment, vintage capital, age-structured systems, optimal control, dynamic programming, Hamilton–Jacobi–Bellman equations, linear convex control, boundary control.

JEL Classification Numbers: C61, C62, E22.

AMS (MOS) subject classification: 49J20, 49J27, 35B37.

1 Introduction

The aim of this paper is to develop the Dynamic Programming (briefly, DP) approach for a family of optimal investment models with vintage capital.

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Optimal investment models with vintage capital[1] have been studied in various papers in the recent years, and modeled differently. That of optimal control of linear age structured equations is one of the possible approaches undertaken in the literature. Such framework has been introduced in in [11, 12] and then studied in various papers, among which we mention [26, 27, 29, 30, 31, 32]. There the optimal investment problem with vintage capital is treated in two main cases:

- In [11, 12, 29, 31] the production function is linear and the representative investor is price taker (corresponding to an objective function which is linear in the capital stock). In this case the value function is linear and the optimal investment strategies (together with the corresponding capital stock trajectories) can be explicitly calculated. Consequently, a deep qualitative analysis of the problem can be performed, including that of the long run behavior of the capital stock.

- In [32] the case when the production function is linear and with large representative investor (which leads to an objective function which is nonlinear in the capital stock). In this case the value function is non linear and the optimal investment strategies cannot be explicitly calculated. In [32] the problem is studied using the Maximum Principle. There the authors recall a particular version of Maximum Principle (first introduced in [30]) and use it to analyze the optimal investment strategies, highlighting in particular an anticipation effect. The paper does not analyze the long run behavior of the capital stock.

We deal with this second case, which is more interesting from the economic point of view but the mathematics is challenging due to the lack of explicit solutions. Indeed the associated optimal control problem is a non standard infinite dimensional problem and cannot be studied with the existing mathematical tools of optimal control. The Maximum Principle approach used in [32] allows to deeply study the optimal investment strategies but appears less efficient in investigating the long run dynamics of the capital stock.

The long run behavior of the capital stock in this nonlinear case is instead our ultimate goal, and the present work represents a first step towards that direction. Indeed, by means of DP we are able to derive a formula for optimal paths and strategies in feedback form (Theorem 5.8), allowing a future study of qualitative properties of optimal couples, moreover we show that the optimal capital path satisfies the Closed Loop Equation (briefly, CLE) which was not yet derived by means, for instance, of Maximum Principle techniques.

Although the present work concerns mainly the theoretical matters, we would like to make clear that it adds both to mathematics and economics: the results contained in Section 5 extend the existing theory of regular solutions of Hamilton–Jacobi–Bellman (HJB from now on) equations in Hilbert spaces to a new set of problems; on the other hand our results are the basis to investigate the properties of the optimal state-control pairs (especially the long run behavior) in our problem (see Section 6) and in those other applications that can be framed into the same setting.

[1] For the study of vintage capital problems we recall also the papers [11, 12, 13, 15, 18, 23].

The paper is organized as follows. In next Section (Section 3) we describe the abstract mathematical problem, the main mathematical difficulties, and the fundamental results. We also review the existing literature on HJB equations in Hilbert spaces.

Then we come to the technical part. In Section 4 we recall the definition of strong solution and the results on existence and uniqueness of strong solutions in the finite horizon case, as they appear in [26]. In Section 5 we study the abstract problem and we state the main results. Proofs are postponed in Appendix A. We end the paper with Section 6 where we apply the results to optimal investment with vintage capital.

2 The optimal investment model with vintage capital

We now describe the model of optimal investment with vintage capital, in the setting introduced by Barucci and Gozzi [11][12], and later reprised and generalized by Feichtinger et al. [30, 31, 32], and by Faggian [26, 27] and Faggian and Gozzi [29].

The capital accumulation process is given by the following system

$$
(2.1) \qquad \begin{cases} \frac{\partial y(\tau,s)}{\partial \tau} + \frac{\partial y(\tau,s)}{\partial s} + \mu y(\tau,s) = u_1(\tau,s), & (\tau,s) \in]t,+\infty[\times]0,\bar{s}] \\ y(\tau,0) = u_0(\tau), & \tau \in]t,+\infty[\\ y(t,s) = x(s), & s \in [0,\bar{s}] \end{cases}
$$

with $t > 0$ the initial time, $\bar{s} \in [0,+\infty]$ the maximal allowed age, and $\tau \in [0,T[$ with horizon $T = +\infty$. The unknown $y(\tau,s)$ represents the amount of capital goods of age s accumulated at time τ, the initial datum is a function $x \in L^2(0,\bar{s})$ (the space of square integrable functions on $(0,\bar{s})$), $\mu > 0$ is a depreciation factor. Moreover, $u_0 : [t,+\infty[\to \mathbb{R}$ is the investment in new capital goods (u_0 is the boundary control) while $u_1 : [t,+\infty[\times[0,\bar{s}] \to \mathbb{R}$ is the investment at time τ in capital goods of age s (hence, the distributed control). Investments are jointly referred to as the control $u = (u_0, u_1)$.

Besides, we consider the firm profits represented by the functional

$$
I(t,x;u_0,u_1) = \int_t^{+\infty} e^{-\lambda\tau}[R(Q(\tau)) - c(u(\tau))]d\tau
$$

where, for some given measurable coefficient α, we have that

$$
Q(\tau) = \int_0^{\bar{s}} \alpha(s) y(\tau,s) ds
$$

is the output rate (linear in $y(\tau)$) R is a concave revenue from $Q(\tau)$ (i.e., from $y(\tau)$). Moreover we have

$$
c(u_0(\tau),u_1(\tau)) = \int_0^{\bar{s}} c_1(s,u_1(\tau,s))ds + c_0(u_0(\tau)),
$$

with c_1 indicating the investment cost rate for technologies of age s, c_0 the investment cost in new technologies, including adjustment-innovation, c_0, c_1 convex in the control variables.

The entrepreneur's problem is that of maximizing $I(t, x; u_0, u_1)$ over all state–control pairs $\{y, (u_0, u_1)\}$ which are solutions (in a suitable sense) of equation (2.1). Such problems are known as *vintage capital* problems, for the capital goods depend jointly on time τ and on age s, which is equivalent to their dependence from time and vintage $\tau - s$.

The mathematical problem that arise in rephrasing optimal investment with vintage capital is an infinite horizon boundary control problem with linear state equation and concave objective function. We are then motivated to study a general family of abstract problems that apply to a variety of examples[2], including optimal investment with vintage capital, and we do so by means of Dynamic Programming.

3 The Mathematical Problem

The abstract problem is the following. Let H and U be separable real Hilbert spaces with scalar products $(\cdot|\cdot)_H$ and $(\cdot|\cdot)_U$ respectively, and we consider a dynamical system of the following type

$$(3.1) \qquad \begin{cases} y'(\tau) = A_0 y(\tau) + Bu(\tau), & \tau \in]t, +\infty[\\ y(t) = x \in H, \end{cases}$$

where H is the state space, $y : [t, +\infty[\to H$ is the trajectory, U is the control space and $u : [t, +\infty[\to U$ is the control, $A_0 : D(A_0) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $\{e^{\tau A_0}\}_{\tau \geq 0}$ on H, and the control operator B is linear and *unbounded*, say $B : U \to [D(A_0^*)]'$. Besides, we consider an *infinite horizon* cost functional given by

$$(3.2) \qquad J_\infty(t, x, u) = \int_t^{+\infty} e^{-\lambda\tau} \left[g_0 \left(y(\tau) \right) + h_0 \left(u(\tau) \right) \right] d\tau$$

where the function g_0 is convex and C^1, and h_0 is *l.s.c.*, convex, superlinear, possibly infinite valued, as better specified later. Our problem is that of minimizing $J_\infty(t, x, u)$ with respect to u over a set $\mathcal{U}$ of admissible controls (which will be denoted with $L^p_\lambda(t, +\infty; U)$, that is an L^p space with a suitable weight, as defined in Section 5).

Then the value function is defined as

$$(3.3) \qquad Z_\infty(t, x) = \inf_{u \in L^p_\lambda(t, +\infty; U)} J_\infty(t, x, u).$$

Since it is easily shown that $Z_\infty(t, x) = e^{-\lambda t} Z_\infty(0, x)$, it is enough to study the HJB equation associated to the problem with initial time $t = 0$, that is

$$(3.4) \qquad -\lambda\psi(x) + (\psi'(x) \mid A_0 x)_H - h_0^*(-B^*\psi'(x)) + g(x) = 0, \; x \in H$$

[2] We also observe that our framework adapts also to other optimal control problems driven by first order PDE's or by delay equations and arising in models of population dynamics (see e.g. [5, 30]), advertising (see e.g. [29, 33, 37, 40]), general equilibrium with vintage capital (see e.g. [15, 23]).

whose candidate solution is $Z_\infty(0,x)$. (Here and in the sequel, h_0^* indicates the Légendre transform of the convex *l.s.c.* function h_0.)

The problem has been already studied by Faggian and by Faggian and Gozzi in the papers [26, 27, 28, 29] in the case of finite horizon, with and without constraints on the control and on the state, yielding a definition of generalized solutions of the associated evolutionary HJB equation. This paper studies instead the infinite horizon case.

Our main results are stated in Section 5, in Theorems 5.6, 5.7, 5.8, where we prove that the value function $Z_\infty(0,\cdot)$ is the unique regular (C^1) solution of the HJB equation (3.4) and that there exists a unique optimal control strategy in feedback form. Moreover the value function is the limit of value functions of suitable finite horizon problems.

We obtain the results by means of the procedure introduced by Barbu and Da Prato [6] (see also Di Blasio [21, 22]) that consists, roughly speaking, in the following steps:

- Consider a family of suitable problems with finite horizon T, with value functions Ψ_T and show they are the unique regular solutions of the corresponding family of evolutionary HJB equations.[3]

- Show that the value functions Ψ_T converge, as $T \to +\infty$, to a regular function Ψ_∞.

- Prove that Ψ_∞ is the unique solution of the stationary HJB equation and that it is equal to the value function, Z_∞, of the infinite horizon problem with initial time $t=0$; prove the existence and uniqueness of optimal feedbacks.

In our (boundary control) case a sharp refinement of this methods is needed. Indeed, with respect to the papers quoted above, our problem features two new nontrivial difficulties:

- The presence of the boundary control yields the unboundedness of the control operator B in the state equation (3.1) and, as a consequence, the discontinuity of the Hamiltonian in the HJB equation (3.4). This fact, coupled with the non-analyticity of the semigroup generated by A, induces us to work in an enlarged space $V' \supset H$. This setting was already introduced in [26, 27] to treat the corresponding finite horizon problem. Of course, since in the examples in Section 6 the parameters have significance only in H, we need to prove that when in the extended setting the initial datum x is in H, then the whole optimal trajectory lies in H, and the optimal control behaves accordingly.

- The running costs g_0 and h_0 are not bounded from below. This means that a two-sided inequality has to be proved in order to show the convergence as $T \to +\infty$. To this extent, we exploit the coercivity of the function h_0 to derive that optimal controls are bounded in L^p_λ.

We end the the subsection with a brief synthesis on the mathematical literature that deal with similar problems.

[3] Indeed these facts in our case were shown in [26, 27], refining the convex regularization method by Barbu and Da Prato contained in [6].

We recall that optimal control problems for infinite dimensional systems and the associated HJB equation have been studied in two different frameworks: one is that of classical and strong solutions, and the other is that of viscosity solutions. We recall also that, as far as we know, verification techniques have been performed in infinite dimension just in the classical/strong context, for they require the value function to be regular (at least in the state variable).

Regarding Dynamic Programming for boundary control problems only few results are available. For the case of linear systems and quadratic costs (where HJB equation reduces to the operator Riccati equation) the reader is referred *e.g.* to the book by Lasiecka and Triggiani [39], to the book by Bensoussan, Da Prato, Delfour and Mitter [14], and, for the case of nonautonomous systems, to the papers by Acquistapace, Flandoli and Terreni [1, 2, 3, 4]. For the case of a linear system and a general convex cost, we mention the papers by Faggian [24, 25, 26, 27, 28], by Faggian and Gozzi [29]. On Pontryagin maximum principle for boundary control problems we mention again the book by Barbu and Precupanu (Chapter 4 in [10]).

For the case of distributed control the literature is indeed richer: we refer the reader to Barbu and Da Prato [6, 7, 8] for some linear convex problems, to Di Blasio [21, 22] for the case of constrained control, to Cannarsa and Di Blasio [16] for the case of state constraints, to Barbu, Da Prato and Popa [9] and to Gozzi [34, 35, 36] for semilinear systems.

For viscosity solutions and HJB equations in infinite dimension we mention the series of papers by Crandall and Lions [19] where also some boundary control problem arises. Moreover, for boundary control we mention the papers by Cannarsa, Gozzi and Soner [17] and by Cannarsa and Tessitore [20] on existence and uniqueness of viscosity solutions of HJB equation. We note also that a verification theorem in the case of viscosity solutions has been proved in some finite dimensional case in [38, 41].

Regarding applications, in addition to the economic literature recalled above, we refer the reader to the many examples contained in the books by Lasiecka and Triggiani [39] and by Bensoussan *et al* [14].

4 Preliminaries: the finite horizon case.

We here recall all the relevant results on the *finite horizon* case that are needed in the sequel. According to the notation in [26], if X and Y are Banach spaces, we set

$$Lip(X;Y) = \{f : X \to Y \ : \ [f]_L := \sup_{x,y\in X,\ x\neq y} \frac{|f(x)-f(y)|_Y}{|x-y|_X} < +\infty\}$$

$$C^1_{Lip}(X) := \{f \in C^1(X) \ : \ [f']_L < +\infty\}$$

$$\mathcal{B}_p(X,Y) := \{f : X \to \mathbb{R} \ : \ |f|_{\mathcal{B}_p} := \sup_{x\in X} \frac{|f(x)|_Y}{1+|x|_X^p} < +\infty\}, \quad \mathcal{B}_p(X) := \mathcal{B}_p(X,\mathbb{R}).$$

Moreover we set

$$\Sigma_0(X) := \{w \in \mathcal{B}_2(X) \ : \ w \text{ is convex, } w \in C^1_{Lip}(X)\}$$

and, for $T > 0$

$$\mathcal{Y}([0,T] \times X) = \{w : [0,T] \times X \to \mathbb{R} \;:\; w \in C([0,T], \mathcal{B}_2(X)),\\ w(t,\cdot) \in \Sigma_0(X),\ \forall t \in [0,T],\quad w_x \in C([0,T], \mathcal{B}_1(X,X'))\}$$

Then we consider two Hilbert spaces V, V', being dual spaces, which we do not identify for reasons which are recalled in Remark 4.2 and we indicate with $\langle \cdot, \cdot \rangle$ the duality pairing. We set V' as the state space of the problem, and denote with U the control space, being U another Hilbert space.

Given an initial time $t \geq 0$ an initial state $x \in V'$, a finite horizon $T > t$, a number $p > 1$, and a control $u \in L^p(t,T;U)$ we consider the trajectory in V'

$$y(\tau) = e^{(\tau - t)A} x + \int_t^\tau e^{(\tau - \sigma)A} B u(\sigma) d\sigma, \quad \tau \in [t,T], \tag{4.1}$$

and a profit functional of type

$$J_T(t,x,u) = \int_t^T \left[g\left(\tau, y(\tau)\right) + h\left(\tau, u(\tau)\right) \right] d\tau + \varphi(y(T)). \tag{4.2}$$

We deal with the problem of minimizing $J_T(t,x,\cdot)$ over all $u \in L^p(t,T;U)$ taking the following set of assumptions on the data.

Assumptions 4.1.
1. $A : D(A) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{\tau A}\}_{\tau \geq 0}$ on V';
2. $B \in L(U, V')$;
3. there exists $\omega_0 \geq 0$ such that $|e^{\tau A} x|_{V'} \leq e^{\omega_0 \tau} |x|_{V'}, \ \forall \tau \geq 0$;
4. $g \in \mathcal{Y}([0,T] \times V')$, $t \mapsto [g_x(t,\cdot)]_L \in L^1(0,T)$
5. $\varphi \in \Sigma_0(V')$;
6. $h(t,\cdot)$ is convex, lower semi–continuous, $\partial_u h(t,\cdot)$ is injective for all $t \in [0,T]$.
7. If is set $\mathcal{H}(t,u) := [h(\tau,\cdot)]^*(u)$, then we assume $\mathcal{H} \in \mathcal{Y}([0,T] \times U)$, $\mathcal{H}(t,0) = 0$, and $\sup_{t \in [0,T]} [\mathcal{H}_u(t,\cdot)]_L < +\infty$.

Remark 4.2. We do not identify V and V' for in the applications the problem is naturally set in a Hilbert space H, such that $V \subset H \equiv H' \subset V'$ (with all bounded inclusions). Indeed, in order to avoid the discontinuities due to the presence of B, as they appear in (3.1)(3.2), we work in the extended state space V' related to H in the following way: V is the Hilbert space $D(A_0^*)$ endowed with the scalar product $(v|w)_V := (v|w)_H + (A_0^* v | A_0^* w)_H$, V' is the dual space of V endowed with the operator norm. Then assume that $B \in L(U, V')$, and extend the semigroup $\{e^{tA_0}\}_{t \geq 0}$ on H to a semigroup $\{e^{tA}\}_{t \geq 0}$ on the space V', having infinitesimal generator A, a proper extension of A_0. The reader is referred to [27] for a detailed treatment. ∎

Remark 4.3. Note that the functions g and ϕ arising from applications usually appear to be defined and C^1 on H, not on the larger space V'. Then, we here need to *assume* that they can be extended to *C^1-regular functions on V'* - which is a non trivial issue. We refer the reader to Section 6 to see how such extension is obtained in the specific case of the economic example, and to [26] and [27] for a thorough discussion of this issue. ∎

Remark 4.4. In Assumption 4.1[7], we assumed $\mathcal{H}(t, 0) = 0$. Such assmption is not restrictive since $\mathcal{H}(t, 0) = -\inf_{v\in U} h(t, v)$ and, if this value is not 0, we may reduce to this case simply setting $\bar{g} = g + \inf_{v\in U} h(t, v)$ and $\bar{h} = h - \inf_{v\in U} h(t, v)$ and treating the problem with $\bar{g}$ and $\bar{h}$ in place of g and h. Note also that the assumption $\partial h(t, \cdot)$ injective is intended to yield a good definition for $\mathcal{H}_u$ as it is, roughly speaking, $\mathcal{H}_u = (\partial h)^{-1}$. Note also that once one has the datum h, its convex conjugate $\mathcal{H}$ is very often explicitly computed. Then the assumptions on $\mathcal{H}$ are essentially assumptions on its convex conjugate h, but more conveniently stated to ensure $\mathcal{H}$ has the desired properties. ∎

Such optimal control problem can be associated by means of dynamic programming, to the following Hamilton-Jacobi-Bellman equation

$$(4.3)\qquad \begin{cases} v_t(t,x) - \mathcal{H}(t, -B^* v_x(t,x)) + \langle Ax | v_x(t,x)\rangle + g(t,x) = 0, & (t,x) \in [0,T] \times V' \\ v(T,x) = \varphi(x), \end{cases}$$

that can be written, by the change if variable $v(t,x) = \phi(T-t, x)$, as

$$(4.4)\qquad \begin{cases} \phi_t(t,x) + \mathcal{H}(T-t, -B^*\phi_x(t,x)) - \langle Ax, \phi_x(t,x)\rangle = g(T-t,x), & (t,x) \in [0,T] \times V' \\ \phi(0,x) = \varphi(x). \end{cases}$$

Finally, the value function of the problem is defined as

$$(4.5)\qquad W_T(t,x) = \inf_{u \in L^p(t,T;U)} J_T(t,x,u),$$

Indeed in [26] Faggian proved existence and uniqueness of strong solutions, as defined shortly afterwards, for a class of more general HJB equations, that is

$$(4.6)\qquad \begin{cases} \phi_t(t,x) + F(t, \phi_x(t,x)) - \langle Ax, \phi_x(t,x)\rangle = g(T-t,x), & (t,x) \in [0,T] \times V' \\ \phi(0,x) = \varphi(x), \end{cases}$$

where F satisfies

$$(4.7)\qquad F \in \mathcal{Y}([0,T] \times V), \quad F(t,0) = 0, \quad \sup_{t\in[0,T]} [F_p(t,\cdot)]_L < +\infty$$

Note indeed that if we set

$$F(t,p) := \mathcal{H}(T-t, -B^*p) = \sup_{u\in U}\{(u| - B^*p)_U - h(T-t,u)\}.$$

then F satisfies (4.7) and it is well defined for p in V, to which $\phi_x(t,x)$ belongs.

Definition 4.5. *Let Assumptions 4.1* $1-5$*, and* (4.7) *be satisfied. We say that* $\phi \in C([0,T],\mathcal{B}_2(V'))$ *is a* strong *solution of* (4.6) *if there exists a family* $\{\phi^\varepsilon\}_\varepsilon \subset C([0,T],\mathcal{B}_2(V'))$ *such that:*

(*i*) $\phi^\varepsilon(t,\cdot) \in C^1_{Lip}(V')$ *and* $\phi^\varepsilon(t,\cdot)$ *is convex for all* $t \in [0,T]$*;* $\phi^\varepsilon(0,x) = \varphi(x)$ *for all* $x \in V'$*.*

(*ii*) *there exist constants* $\Gamma_1, \Gamma_2 > 0$ *such that*

$$\sup_{t\in[0,T]} [\phi^\varepsilon_x(t)]_L \le \Gamma_1, \quad \sup_{t\in[0,T]} |\phi^\varepsilon_x(t,0)|_V \le \Gamma_2, \ \forall \varepsilon > 0;$$

(*iii*) *for all* $x \in D(A)$*,* $t \mapsto \phi^\varepsilon(t,x)$ *is continuously differentiable;*

(*iv*) $\phi^\varepsilon \to \phi$*, as* $\varepsilon \to 0+$*, in* $C([0,T],\mathcal{B}_2(V'))$*;*

(*v*) *there exists* $g_\varepsilon \in C([0,T];\mathcal{B}_2(V'))$ *such that, for all* $t \in [0,T]$ *and* $x \in D(A)$*,*

$$\phi^\varepsilon_t(t,x) - F(t,\phi^\varepsilon_x(t,x)) + \langle Ax, \phi^\varepsilon_x(t,x)\rangle = g_\varepsilon(T-t,x)$$

with $g_\varepsilon(t,x) \to g(t,x)$*, and* $\int_0^T |g_\varepsilon(s) - g(s)|_{C_2} ds \to 0$*, as* $\varepsilon \to 0+$ *.*

The main result contained in [26] is the following.

Theorem 4.6. *Let Assumptions 4.1* $1-5$*, and* (4.6) *be satisfied. There exists a unique strong solution* ϕ *of* (4.4) *in the class* $C([0,T],\mathcal{B}_2(V'))$ *with the following properties:*

(*i*) *for all* $x \in D(A)$*,* $\phi(\cdot,x)$ *is Lipschitz continuous;*

(*ii*) $\phi \in \mathcal{Y}([0,T]\times V')$*. Moreover the following estimate is satisfied for all* $t \in [0,T]$

$$[\phi_x(t)]_L \le e^{2\omega_0 t}[\varphi']_L + \int_0^t e^{2\omega_0(t-s)}[g_x(T-s,\cdot)]_L ds. \tag{4.8}$$

Regarding applications to the optimal control problem, in [27] we were able to prove what follows.

Theorem 4.7. *Let Assumptions 4.1* $1-7$ *be satisfied, and let* ϕ *be the strong solution of* (4.4) *described in Theorem 4.6. Then*

$$W_T(t,x) = \phi(T-t,x), \ \forall t \in [0,T], \ \forall x \in V',$$

that is, the value function W_T *of the optimal control problem is the unique strong solution of the backward HJB equation* (4.3)*.*

5 The infinite horizon problem

We describe the abstract setup of the infinite horizon optimal control problem and we state the main result of the paper, namely Theorem 5.7, that establishes that the value function of our problem is the unique regular solution of the associated HJB equation. Some other important results follow, such as Theorem 5.8 on existence and uniqueness of optimal feedbacks, and Theorem 5.6, establishing the connection between finite and infinite horizon value functions. Proofs of all assertions are found in section A.

We use the same framework as that in Section 4, for the finite horizon problem. As one expects, the state space is V' and the control space is U. The state equation is given in V' as

$$y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}Bu(\sigma)d\sigma, \quad \tau \in [t, +\infty[, \tag{5.1}$$

while, for all $x \in V'$ and $t > 0$, the target functional is of type

$$J_\infty(t, x, u) := \int_t^{+\infty} e^{-\lambda\tau}[g_0(y(\tau)) + h_0(u(\tau))]d\tau. \tag{5.2}$$

We assume the following hypotheses:

Assumptions 5.1. 1. $A : D(A) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{\tau A}\}_{\tau\geq 0}$ on V';

2. $B \in L(U, V')$;

3. there exists $\omega \in \mathbb{R}$ such that $|e^{\tau A}x|_{V'} \leq e^{\omega\tau}|x|_{V'}$, $\forall \tau \geq 0$;

4. $g_0, \phi_0 \in \Sigma_0(V')$

5. h_0 is convex, lower semi–continuous, $\partial_u h_0$ is injective.

6. $h_0^*(0) = 0$, $h_0^* \in \Sigma_0(V)$;

7. $\exists a > 0$, $\exists b \in \mathbb{R}$, $\exists p > 1 :\ h_0(u) \geq a|u|_U^p + b$, $\forall u \in U$;

   Moreover, either

8.a $p > 2$, $\lambda > (2\omega \vee \omega)$.

   or

8.b $\lambda > \omega$, and $g_0, \phi_0 \in \mathcal{B}_1(V')$.

Remark 5.2. Note that the Assumption 5.1 [3] above implies that also Assumption 4.1 [3] where $\omega_0 = \omega \vee 0$.

The functional $J_\infty(t; x, u)$ has to be minimized with respect to u over the set of admissible controls

$$L^p_\lambda(t, +\infty; U) = \{u \in L^1_{loc}(t, +\infty; U)\ ;\ t \mapsto u(t)e^{-\frac{\lambda t}{p}} \in L^p(t, +\infty; U)\}, \tag{5.3}$$

which is Banach space with the norm

$$\|u\|_{L^p_\lambda(t,+\infty;U)} = \int_t^{+\infty} |u(\tau)|_U^p e^{-\lambda\tau}d\tau = \|e^{-\frac{\lambda(\cdot)}{p}}u\|_{L^p(t,+\infty;U)}.$$

Similarly, the space $L^p_\lambda(t,s;U)$ endowed with the norm

$$\|u\|_{L^p_\lambda(t,s;U)} = \int_t^s |u(\tau)|^p_U e^{-\lambda\tau} d\tau = \|e^{-\frac{\lambda(\cdot)}{p}} u\|_{L^p(t,s;U)}.$$

is a Banach space. Then (5.3) is the natural set of admissible controls to get estimates in this setting (see e.g Lemma A.5 and Lemma A.7).

The value function is then defined as

$$Z_\infty(t,x) = \inf_{u \in L^p_\lambda(t,+\infty;U)} J_\infty(t,x,u).$$

As it is easy to check that

$$Z_\infty(t,x) = e^{-\lambda t} Z_\infty(0,x)$$

one may associate to the problem the following stationary HJB equation

$$-\lambda\psi(x) + \langle \psi'(x), Ax\rangle - h_0^*(-B^*\psi'(x)) + g(x) = 0, \tag{5.4}$$

whose candidate solution is the function $Z_\infty(0,\cdot)$.

We will use the following definition of solution for equation (5.4).

Definition 5.3. *A function ψ is a classical solution of the stationary HJB equation* (5.4) *if it belongs to $\Sigma_0(V')$ and satisfies* (5.4) *pointwise for every $x \in D(A)$.*

Remark 5.4. The reader has certainly realized that Assumptions 5.1 $[1-7]$ imply Assumptions 4.1 $[1-7]$. Moreover, as mentioned thoroughly in Remark 4.3, we need to assume that the functions g_0 and ϕ_0 can be extended to C^1-regular functions on V'. ■

Remark 5.5. See Remark 4.4 for some comments on h_0 and h_0^* that apply also to this case. ■

Before proving that the value function of the infinite horizon problem starting at $(0,x)$, namely $Z_\infty(0,x)$, is the unique classical solution to the stationary HJB equation, some preliminary work is needed. First we show that $Z_\infty(0,x)$ is the limit as t tends to $+\infty$ of a suitable family of value functions for finite horizon, along with their gradients. Doing so, we also establish that Z_∞ inherits from that family the C^1 regularity in x which we need to solve the stationary HJB equation, and which is so precious when building optimal feedback maps.

Theorem 5.6. *Let Assumptions 5.1 be satisfied. Let also $\phi_T(t,x)$ be the unique strong solution to* (4.4)*. Then the function*

$$\Psi(t,x) := e^{\lambda(T-t)}\phi_T(t,x)$$

is independent of T and there exists the following limit

$$\Psi_\infty(x) := \lim_{t\to+\infty} \Psi(t,x).$$

The convergence is uniform on bounded subsets of V'. Moreover, if $\lambda > \omega \max\{2, \frac{p}{p-1}\}$, then $\Psi_\infty \in \Sigma_0(V')$. Moreover, for every fixed $x \in V'$

$$\Psi_x(t,x) \to \Psi'_\infty(x), \text{ weakly in } V, \text{ as } t \to +\infty.$$

Hence, Ψ_∞ being the candidate solution to the stationary HJB equation (5.4), one shows what follows.

Theorem 5.7. *Let Assumptions 5.1 hold. Then:*

(i) Ψ_∞ *is the value function of the infinite horizon problem with initial time* $t=0$*, that is*

$$\Psi_\infty(x) = Z_\infty(0,x) = \inf_{u \in L^p_\lambda(0,+\infty;U)} J_\infty(0,x,u).$$

Moreover $Z_\infty(t,x) = e^{-\lambda t}\Psi_\infty(x)$*;*

(ii) Ψ_∞ *is a classical solution (as defined in Definition 5.3) of the stationary Hamilton-Jacobi-Bellman equation* (5.4)*. that is*

$$-\lambda \Psi_\infty(x) + \langle \Psi'_\infty(x), Ax\rangle - h_0^*(-B^*\Psi'_\infty(x)) + g(x) = 0.$$

(iii) The function Ψ_∞ *is the unique classical solution to* (5.4)*.*

Once we have established that Ψ_∞ is the classical solution to the stationary HJB equation, and that it is differentiable, we can build optimal feedbacks and prove the following theorem.

Theorem 5.8. *Let Assumptions 5.1 hold. Let* $t \geq 0$ *and* $x \in V'$ *be fixed. Then there exists a unique optimal pair* (u^*, y^*)*. The optimal state* y^* *is the unique solution of the Closed Loop Equation*

$$y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}B(h_0^*)'(-B^*\Psi'_\infty(y(s)))d\sigma, \quad \tau \in [t,+\infty[. \tag{5.5}$$

while the optimal control u^* *is given by the feedback formula*

$$u^*(s) = (h_0^*)'(-B^*\Psi'_\infty(y^*(s))).$$

6 The economic example of optimal investment with vintage capital

When rephrased in an infinite dimensional setting, with $H := L^2(0,\bar{s})$ as state space, the state equation (2.1) can be reformulated as a linear control system with an unbounded control operator, that is

$$\begin{cases} y'(\tau) = A_0 y(\tau) + Bu(\tau), & \tau \in]t,+\infty[; \\ y(t) = x, \end{cases} \tag{6.1}$$

where $y : [t,+\infty[\to H$, $x \in H$, $A_0 : D(A_0) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{A_0 t}\}_{t\geq 0}$ on H with domain $D(A_0) = \{f \in H^1(0,\bar{s}) : f(0) = 0\}$ and defined as $A_0 f(s) = -f'(s) - \mu f(s)$, the control space is $U = \mathbb{R} \times H$, the control function is a couple $u \equiv (u_0, u_1) : [t, +\infty[\to \mathbb{R} \times H$, and the control operator is given by $Bu \equiv B(u_0, u_1) = u_1 + u_0\delta_0$, for all $(u_0, u_1) \in \mathbb{R} \times H$, δ_0 being the Dirac delta at the point 0. Note that, although $B \notin L(U, H)$, is $B \in L(U, D(A_0^*)')$. The reader can find in [11]the (simple) proof of the following theorem, which we will exploit in a short while.

Theorem 6.1. *Given any initial datum $x \in H$ and control $u \in L^p_\lambda(t, +\infty; U)$ the mild solution of the equation (6.1)*

$$y(s) = e^{(s-t)A}x + \int_t^s e^{(s-\tau)A}Bu(\tau)d\tau$$

belongs to $C([t, +\infty); H)$.

Following Remark 4.2, we then set

$$V = D(A_0^*) = \{f \in H^1(0, \bar{s}) : f(\bar{s}) = 0\}$$

and $V' = D(A_0^*)'$. Regarding the target functional, we set

$$J_\infty(t, x; u) := -I(t, x; u_0, u_1),$$

with:
$g_0 : V' \to \mathbb{R},\ g_0(x) = -R(\langle \alpha, x\rangle)$,
$h_0 : U \to \mathbb{R},\ h_0(u) = c_0(u_0) + \int_0^{\bar{s}} c_1(s, u_1(s))ds$.

Remark 6.2. As announced in Remark 5.4, here the extension of the datum g_0 to V' is straightforward, as long as we assume that $\alpha \in V$ and replace scalar product in H with the duality in V, V'.

Note further that $\omega = 0$, $\lambda > 0$ (the type of the semigroup is negative and equal to $-\mu$). ∎

As the problem now fits into our abstract setting, the main results of the previous sections apply to the economic problem when data R, c_0, c_1 satisfy Assumption 5.1[8.a] or [8.b]. In particular, such thing happens in the following two interesting cases:

- If we assume, for instance, that R is a concave, C^1, sublinear function (for example one could take R quadratic in a bounded set and then take its linear continuation, see e.g. [30, 32]), and c_0, c_1 quadratic functions of the control variable, then Assumption 5.1[8.b] holds.

- Assumption 5.1[8.a] is instead satisfied when R is, for instance, quadratic - as it occurs in some other meaningful economic problems - and c_0, c_1 are equal to $+\infty$ outside some compact interval, and equal to any convex *l.s.c.* function otherwise. Such case corresponds to that of constrained controls (controls that violate the constrain yield infinite costs).

In these cases, Theorems 5.6, 5.7, 5.8 hold true. In particular Theorem 5.8 states the existence of a unique optimal pair (u^*, y^*) for any initial datum $x \in V'$. Note that in general the optimal trajectory y^* lives in V'. However, since the economic problem makes sense in H, we would now like to infer that whenever x (the initial age distribution of capital) lies in H, then the whole optimal trajectory lives in H. Indeed, this is guaranteed by Theorem 6.1.

All these results allow to perform the analysis of the behavior of the optimal pairs and to study phenomena such as the diffusion of new technologies (see e.g. [11, 12]) and the anticipation effects (see e.g. [30, 32]). With respect to the results in [11, 12], here also the case of nonlinear R (which is particularly interesting from the economic point of view, as it takes into account the case of large investors) is considered. With respect to the results in [30, 32], here the existence of optimal feedbacks yields a tool to study more deeply the long run behavior of the trajectories, like the presence of long run equilibrium points and their properties.

7 Conclusion

In this paper we have considered an optimal investment model with vintage capital where the revenue function R is nonlinear. This is motivated e.g. by the study of the case of large representative investors. We have embedded the problem in a class of optimal control problems in infinite dimension that has not been treated so far in the literature since it contains various nontrivial technical difficulties to overcome. Using the Dynamic Programming approach we have proven that the value function is the unique solution of the associated HJB equation and, consequently, the existence of optimal feedback controls. We have proved that such results apply to our vintage capital problem and observed that this provide a solid basis to study the long run behavior of the optimal capital trajectory. This will require additional nontrivial work, due to the infinite dimensionality of the problem and will be done in a subsequent paper.

A Proofs of the main results

In this section we prove the theorems stated in Section 5.

A.1 Auxiliary functions, equations and estimates

We study infinite horizon by means of finite horizon. Then it is worth noting that, thanks to the particular dependence of data on the time variable, we can associate to the HJB equation arising in finite horizon the following equation:

$$\text{(A.1)} \qquad \begin{cases} z_t(t,x) - \lambda z(t,x) + \langle Ax, z_x(t,x)\rangle - h_0^*(-B^* z_x(t,x)) + g_0(x) = 0 \\ z(T,x) = \phi_0(x) \end{cases}$$

and define a strong solution of (A.1) as follows.

Definition A.1. *Let $(t, x) \in [0, T] \times V'$. We say that Z_T is a strong solution to* (A.1) *if*

$$Z_T(t, x) = e^{\lambda t} v_T(t, x)$$

with v_T any strong solution to (4.3)*, in the sense of Definition 4.5.*

Remark A.2. Equation (A.1) is obtained formally from (4.3) with the change of variable $v(t, x) = e^{-\lambda t} z(t, x)$. Note that one could give a direct definition of solution of (A.1) (without passing through strong solutions of (4.3)) in the spirit of Definition (4.5). ∎

Note that

$$u \in L^p(t, T; U) \iff u \in L^p_\lambda(t, T; U)$$

so that all minimization procedure in Section 2 can be equivalently operated in $L^p(t, T; U)$ or in $L^p_\lambda(t, T; U)$. Then, recalling that the unique strong solution to (4.3) is the value function of the optimal control problem (see [27]), the following result is readily proven.

Proposition A.3. *Let Assumptions 4.1 be satisfied, and let Z_T be the unique strong solution to* (A.1)*. Then*

$$Z_T(t, x) = \inf_{u \in L^p_\lambda(t,T;U)} \left\{ \int_t^T e^{-\lambda(\tau - t)} [g_0(y(\tau)) + h_0(u(\tau))] d\tau + e^{-\lambda(T-t)} \phi_0(y(T)) \right\}. \tag{A.2}$$

We may also write a forward version of $(A.1)$, that is

$$\begin{cases} \psi_t(t, x) + \lambda \psi(t, x) - \langle Ax, \psi_x(t, x) \rangle + h_0^*(-B^* \psi_x(t, x)) = g_0(x) \\ \psi(0, x) = \phi_0(x) \end{cases} \tag{A.3}$$

with $(t, x) \in [0, T] \times V'$ and $\psi(t, x) = z(T - t, x)$, where Z is the unique strong solution of $(A.1)$, and then prove the following important result.

Lemma A.4. *Let $\Psi_T(t, x) = Z_T(T - t, x)$, where Z is given by* (A.2)*. Then Ψ does not depend on T, that is*

$$\Psi(t, x) \equiv \Psi_T(t, x) = \inf_{u \in L^p_\lambda(0,t;U)} \left\{ \int_0^t e^{-\lambda \tau} [g_0(y(\tau)) + h_0(u(\tau))] d\tau + e^{-\lambda t} \phi_0(y(t)) \right\}. \tag{A.4}$$

Moreover a Dynamic Programming Principle holds

$$\Psi(t, x) = \inf_{u \in L^p_\lambda(0,s;U)} \left\{ \int_0^s e^{-\lambda \tau} [g_0(y(\tau)) + h_0(u(\tau))] d\tau + e^{-\lambda s} \Psi(t - s, y(s)) \right\}, \ \forall s \in [0, t]. \tag{A.5}$$

Proof. By changing the variable, for all $0 < s < t$, and $u \in L^p_\lambda(s, t; U)$, we have

$$J_t(s, x, u(\cdot)) = e^{-\lambda s} J_{t-s}(0, x, u(\cdot + s)). \tag{A.6}$$

Then by definition of value function and $(A.2)$, we have

$$\Psi_T(t, x) = \inf_{u \in L^p_\lambda(T-t,T;U)} J_T(T - t, x, u) = \inf_{\bar{u} \in L^p_\lambda(0,t;U)} J_t(0, x, \bar{u})$$

where the last equality is obtained by setting $\bar{u}(s) := u(s+T-t)$, and observing that

$$u \in L^p_\lambda(T-t, T; U) \iff \bar{u} \in L^p_\lambda(0, t; U).$$

Then (A.4) follows by generality of $\bar{u}$.

The proof of the Dynamic Programming Principle is standard and we omit it. ∎

Here follow some other technical results that will be frequently exploited in the sequel.

Lemma A.5. *In Assumptions 5.1, if $u \in L^p_\lambda(t, T; U)$ is an admissible control and $y(\tau) \equiv y(\tau; t, x, u)$ is the associated trajectory, and $q = p/(p-1)$, then for suitable positive constant C independent of t and x the following estimates hold:*

$$\int_t^s e^{-\omega\tau} |u(\tau)|_U d\tau \leq \theta(t,s)^{\frac{1}{q}} \|u\|_{L^p_\lambda(t,s;U)} \tag{A.7}$$

$$|y(\tau)|_{V'} \leq C e^{\omega\tau} \left[|x|_{V'} + \theta(t,\tau)^{\frac{1}{q}} \|u\|_{L^p_\lambda(t,\tau;U)} \right] \tag{A.8}$$

$$\int_t^s e^{-\lambda\tau} |y(\tau)|_{V'} d\tau \leq C \left[|x|_{V'} + \|u\|_{L^p_\lambda(t,s;U)} + 1 \right], \ \forall s \geq t \tag{A.9}$$

with

$$\theta(t,s) = \begin{cases} \frac{p-1}{|\lambda - p\omega|} \left| e^{q\left(\frac{\lambda}{p} - \omega\right)t} - e^{q\left(\frac{\lambda}{p} - \omega\right)s} \right| & \lambda \neq \omega p \\ |t-s| & \lambda = \omega p. \end{cases}$$

Remark A.6. Note that all inequalities in the following proof hold also for $p = 2$, but the main results in section 5 require also $p > 2$. Moreover, in the set of Assumptions 5.1

$$\omega < 0 \implies \lambda > \omega p.$$

Indeed, from [8.b] follows $p > 1$ which implies $\{\lambda \ : \ \omega < \lambda < p\omega\} = \emptyset$, while from [8.a] follows $p \geq 2$ which implies $\{\lambda \ : \ 2\omega < \lambda < p\omega\} = \emptyset$. ∎

Proof. In what follows we denote by C a positive constant not depending on t, x and u. Inequality (A.7) holds by means Hölder's inequality. From (A.7) follows

$$\begin{aligned} |y(\tau)|_{V'} &\leq \max\{\|B\|_{L(U,V')}, e^{|\omega|t}\} e^{\omega\tau} \left(|x|_{V'} + \int_t^\tau e^{-\omega\sigma} |u(\sigma)|_U d\sigma \right) \\ &\leq C e^{\omega\tau} \left[|x|_{V'} + \theta(t,\tau)^{\frac{1}{q}} \|u\|_{L^p_\lambda(t,\tau;U)} \right], \end{aligned} \tag{A.10}$$

so that also (A.8) is proven. To prove (A.9) we need to estimate the right hand side in

$$\int_t^s e^{-\lambda\tau} |y(\tau)|_{V'} \leq \frac{C}{\lambda}(e^{-\lambda t} - e^{-\lambda s}) |x|_{V'} + C \|u\|_{L^p_\lambda(t,s;U)} \int_t^s e^{-(\lambda - \omega)\tau} \theta(t,\tau)^{\frac{1}{q}} d\tau \tag{A.11}$$

Indeed, in case $\lambda > \omega p$, one derives

$$
\begin{aligned}
e^{-(\lambda-\omega)\tau}\theta(t,\tau)^{\frac{1}{q}} &\le Ce^{-(\lambda-\omega)\tau}\left[e^{q\left(\frac{\lambda}{p}-\omega\right)\tau}-e^{q\left(\frac{\lambda}{p}-\omega\right)t}\right]^{\frac{1}{q}} \\
&\le Ce^{-\frac{\lambda}{q}\tau},
\end{aligned}
\tag{A.12}
$$

while similarly in case $\lambda < \omega p$, one obtains

$$
e^{-(\lambda-\omega)\tau}\theta(t,\tau)^{\frac{1}{q}} \le Ce^{-(\lambda-\omega)\tau}e^{\left(\frac{\lambda}{p}-\omega\right)t} \le Ce^{-(\lambda-\omega)\tau}
$$

(we recall that in such case $\lambda-\omega>0$ in view of Remark A.6). Hence, when $\lambda \neq \omega p$, we have

$$
\int_t^s e^{-(\lambda-\omega)\tau}\theta(t,\tau)^{\frac{1}{q}}d\tau \le C\|u\|_{L^p_\lambda(t,s;U)}\; e^{-\left[\frac{\lambda}{q}\wedge(\lambda-\omega)\right]t},
\tag{A.13}
$$

for all s. In the case $\lambda=\omega p$, on the other hand, there exists $\delta>0$ such that $\lambda-\omega-\delta>0$, and consequently $T_\delta \ge t$ such that

$$
e^{-\delta\tau}|\tau-t|^{\frac{1}{q}} \le 1,\ \forall \tau \ge T_\delta.
$$

Then

$$
\begin{aligned}
\int_t^s e^{-(\lambda-\omega)\tau}\theta(t,\tau)^{\frac{1}{q}}d\tau &\le \int_t^{+\infty} e^{-(\lambda-\omega)\tau}\theta(t,\tau)^{\frac{1}{q}}d\tau \\
&\le |T_\delta-t|^{\frac{1}{q}}\int_t^{T_\delta} e^{-(\lambda-\omega)\tau}d\tau + \int_{T_\delta}^{+\infty} e^{-(\lambda-\omega-\delta)\tau}d\tau \\
&\le \frac{1}{\lambda-\omega}|T_\delta-t|^{\frac{1}{q}}(e^{-(\lambda-\omega)t}-e^{-(\lambda-\omega)T_\delta}) + \frac{1}{\lambda-\omega-\delta}e^{-(\lambda-\omega-\delta)T_\delta} \\
&\le C
\end{aligned}
\tag{A.14}
$$

for a suitable constant C. Then applying (A.14) and (A.13) to (A.11) one derives (A.9). ∎

Lemma A.7. *Let Assumptions 5.1 be satisfied. Let $\varepsilon \in [0,1]$ be fixed, $u_\varepsilon \in L^p_\lambda(t,s;U)$ be any ε-optimal control at (t,x), with horizon s for the functional $J_s(t,x,\cdot)$ defined in* (5.2). *Then, for a suitable positive constant K, independent of t, s and x, we have:*
(i) $\|u_\varepsilon\|_{L^p_\lambda(t,s;U)} \le K(1+|x|^2_{V'})$, *when Assumptions 5.1[8.a] holds;*
(ii) $\|u_\varepsilon\|_{L^p_\lambda(t,s;U)} \le K(1+|x|_{V'})$, *when Assumptions 5.1[8.b] holds.*

Proof. Let $\bar u \in dom(h_0)$, and $\bar u(\tau) \equiv \bar u$. Let also $(u_\varepsilon, y_\varepsilon)$ be ε-optimal at (t,x). Then

$$
J_s(t,x,u_\varepsilon)-\varepsilon \le J_s(t,x,\bar u).
\tag{A.15}
$$

On one hand, from the convexity of g_0 and ϕ_0, and from (A.9), there exists some positive constant C_0 such that

$$
\begin{aligned}
J_s(t,x,u_\varepsilon) &\ge \int_t^s e^{-\lambda\tau}(a|u_\varepsilon(\tau)|^p_U+b)d\tau - C_0\int_t^s e^{-\lambda\tau}(1+|y_\varepsilon(\tau)|_{V'})d\tau + \\
&\qquad - C_0e^{-\lambda s}(1+|y_\varepsilon(s)|_{V'}) \\
&\ge a\|u_\varepsilon\|^p_{L^p_\lambda(t,s;U)} + \frac{b-C_0}{\lambda} - C(|x|_{V'}+\|u_\varepsilon\|_{L^p_\lambda(t,s;U)}+1) + \\
&\qquad - C_0e^{-\lambda s} - \ CC_0e^{-(\lambda-\omega)s}(|x|+\|u_\varepsilon\|_{L^p_\lambda(t,s;U)}\theta(t,s)^{\frac{1}{q}})
\end{aligned}
\tag{A.16}
$$

where in the last estimate we applied the assumptions on h_0, and estimates (A.8) (A.9). Since $e^{-(\lambda-\omega)s}\theta(t,s)^{\frac{1}{q}}$ is bounded for all s, the latter implies

$$J_s(t,x,u_\varepsilon) \geq a\|u_\varepsilon\|^p_{L^p_\lambda(t,s;U)} - \gamma_1\|u_\varepsilon\|_{L^p_\lambda(t,s;U)} - \gamma_2|x|_{V'} + \gamma_3 \tag{A.17}$$

for a suitable choice of the constants $\gamma_1, \gamma_2, \gamma_3$. On the other hand, also $J_s(t,x,\bar{u})$ can be estimated by means of either Assumptions 5.1[8.a] or [8.b]. Indeed, we derive that the trajectory $\bar{y}(\tau) = y(t,x,\bar{u})$ satisfies

$$|\bar{y}(\tau)|_{V'} \leq K_1 e^{\omega\tau}(1 + |x|_{V'}),$$

where $K_1 = e^{-\omega t}(1 \vee \|B\||\bar{u}|\omega^{-1})$. Then, if [8.a] holds, $|\bar{y}(\tau)|_{V'} \leq 2K_1^2 e^{2\omega\tau}(1 + |x|^2_{V'})$, so that

$$\begin{aligned} J_s(t,x,\bar{u}) &\leq (|h_0(\bar{u})| + |g_0|_{\mathcal{B}_2})\lambda^{-1} + 2|g_0|_{\mathcal{B}_2}K_1^2(1 + |x|^2_{V'})(\lambda - 2\omega)^{-1} + \\ &+ |\phi_0|_{\mathcal{B}_2}e^{-\lambda s}(1 + 2K_1^2 e^{2\omega\tau}(1 + |x|^2_{V'})) \\ &\leq \gamma_4(1 + |x|^2_{V'}) \end{aligned} \tag{A.18}$$

for a suitable constant γ_4. Hence, by means of (A.15), (A.17) and (A.18), we obtain

$$\|u_\varepsilon\|^p_{L^p_\lambda(t,s;U)}(a\|u_\varepsilon\|^p_{L^p_\lambda(t,s;U)} - \gamma_1) \leq (2\gamma_2 + \gamma_4)(1 + |x|^2) - \gamma_3 + \varepsilon$$

which imply (i). If instead [8.b] holds, then by a similar reasoning one derives

$$J_s(t,x,\bar{u}) \ \leq \gamma_5(1 + |x|_{V'}) \tag{A.19}$$

for a suitable constant γ_5, and then (ii). ∎

Lemma A.8. *Let Assumptions 5.1 be satisfied. If Assumptions 5.1 hold with [8.a]), then* Ψ *satisfies*

$$\exists C > 0 \quad : \quad |\Psi(t,x)| \leq C(1 + |x|^2_{V'}), \ \forall (t,x) \in [0, +\infty[\times V'.$$

If Assumptions 5.1 hold with [8.b]), then Ψ *satisfies*

$$\exists C > 0 \quad : \quad |\Psi(t,x)| \leq C(1 + |x|_{V'}), \ \forall (t,x) \in [0, +\infty[\times V'.$$

Proof. Let $\varepsilon > 0$ be fixed and u_ε be an admissible control, with $y_\varepsilon(\tau) = y(\tau; 0, x, u_\varepsilon)$ the associated trajectory, such that

$$\Psi(t,x) \geq \int_0^t e^{-\lambda\tau}\big[g_0(y_\varepsilon(\tau)) + h_0(u_\varepsilon(\tau))\big]d\tau + e^{-\lambda t}\phi_0(y_\varepsilon(t)) - \varepsilon.$$

Hence from the convexity of g_0 and h_0, for a suitable positive constant γ, and by applying $(A.9)$ we derive

$$\begin{aligned} \Psi(t,x) &\geq -\gamma\int_0^t e^{-\lambda\tau}\Big[1 + |y_\varepsilon(\tau)|_{V'} + |u_\varepsilon(\tau)|_U\Big]d\tau - \varepsilon \\ &\geq -\frac{\gamma}{\lambda} - \gamma C\big[|x|_{V'} + \|u_\varepsilon\|_{L^p_\lambda(0,t;U)} + 1\big] - \gamma|1 - e^{-\lambda t}|^{\frac{1}{q}}\lambda^{-\frac{1}{q}}\|u_\varepsilon\|_{L^p_\lambda(0,t;U)} - \varepsilon \end{aligned}$$

so that by means of Lemma A.7

$$-\Psi(t,x) \le C(1+|x|_{V'}^2)$$

when Assumptions 5.1[8.a] holds, and

$$-\Psi(t,x) \le C(1+|x|_{V'}),$$

when Assumptions 5.1[8.b] holds, for a suitable choice of the constant C. The missing inequality derives from

$$\Psi(t,x) \le J_t(0,x,\bar{u})$$

with $\bar{u}(\tau) = \bar{u} \in dom(h_0)$, when we apply (A.18) when [8.a] holds, or (A.19) when [8.b] holds. ∎

Lemma A.9. *Let Assumptions 5.1 hold, and* $\lambda > \omega \max\{2,q\}$. *Then*

(*i*) $\sup_{t\ge 0}[\Psi_x(t)]_L < +\infty$*;*

(*ii*) $\sup_{t\ge 0} |\Psi_x(t,0)|_V < +\infty$.

Proof. We use estimate (4.8) with $g(s,x) = e^{-\lambda s} g_0(x)$, and $\varphi(x) = e^{-\lambda T}\phi_0(x)$ to derive

$$[\phi_x(t)]_L \le e^{2\omega t} e^{-\lambda T}\left([\phi_0']_L + \frac{[g_0']_L}{\lambda-\omega}\left(e^{(\lambda-\omega)t}-1\right)\right),$$

so that

$$\begin{aligned}
[\Psi_x(t)]_L &= e^{\lambda(T-t)}[\phi_x(t)]_L \\
&\le e^{-(\lambda-2\omega)t}[\phi_0']_L + \frac{[g_0']_L}{\lambda-\omega}(1-e^{-(\lambda-2\omega)t}) \\
&\le [\phi_0']_L + \frac{[g_0']_L}{\lambda-\omega},
\end{aligned}$$

for all $t \ge 0$.

Next we prove (*ii*). Let h be a real number $|h| \le 1$ and $z \in V'$ such that $|z|_{V'} \le 1$. Let u_ε is ε-optimal at $(0,0)$ with horizon t, $y_{0,\varepsilon}(s) := y(s;0,0,u_\varepsilon)$ and $y_{h,\varepsilon}(s) := y(s;0,hz,u_\varepsilon)$, then by means of (A.8) one has

$$|y_{0,\varepsilon}(t)|_{V'} \le \|B\|_{L(U,V')} e^{\omega t}\theta(0,t)^{\frac{1}{q}}\|u_\varepsilon\|_{L^p(0,t;U)},$$

so that

(A.20)
$$
\begin{aligned}
&\frac{\Psi(t,hz)-\Psi(t,0)}{h} \leq \\
&\qquad \leq \int_0^t e^{-\lambda s}[g_0(y_{h,\varepsilon}(s))-g_0(y_{0,\varepsilon}(s))]ds + e^{-\lambda t}[\phi_0(y_{h,\varepsilon}(t))-\phi_0(y_{0,\varepsilon}(t))]+\varepsilon \\
&\qquad \leq \int_0^t e^{-\lambda s}\langle g_0'(y_{0,\varepsilon}(s)), e^{\omega s}|z|_{V'}\rangle_{V'} ds + e^{\lambda t}\langle \phi_0'(y_{0,\varepsilon}(t)), e^{\omega t}|z|_{V'}\rangle_{V'} - \varepsilon \\
&\qquad \leq |z|_{V'}\left[|g_0'|_{B_1}\int_0^t e^{-(\lambda-\omega)s}\left(1+|y_{0,\varepsilon}(s)|_{V'}\right)ds + e^{-(\lambda-\omega)t}|\phi_0'|_{B_1}(1+|y_{0,\varepsilon}(t)|_{V'})\right] - \varepsilon \\
&\qquad \leq |z|_{V'}\left[|g_0'|_{B_1}\left(\frac{1}{\lambda}+\|B\|\|u_\varepsilon\|_{L^p_\lambda(0,t;U)}\int_0^t e^{-(\lambda-2\omega)s}\theta(0,s)^{\frac{1}{q}}ds\right)+\right. \\
&\qquad\qquad \left. + |\phi_0'|_{B_1}(e^{-(\lambda-\omega)t}+e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{1}{q}})\right] - \varepsilon
\end{aligned}
$$

Recalling Lemma A.7, using that $\lambda > q\omega$ and reasoning like in (A.13) and (A.14), one obtains that the following quantities

$$
\|u_\varepsilon\|_{L^p_\lambda(0,t;U)}, \quad e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{1}{q}}, \quad \int_0^t e^{-(\lambda-2\omega)s}\theta(0,s)^{\frac{1}{q}}ds
$$

are bounded by a constant (independent of t), by passing to limits as $h \to 0$ in the preceding inequality one derives

$$
\sup_{t\geq 0}\langle \Psi_x(t,0), z\rangle < +\infty.
$$

On the other hand, by a similar reasoning, and with $u_{h,\varepsilon}$ ε-optimal at $(0,hz)$ with horizon t, there exists a positive constant η such that

$$
\frac{\Psi(t,0)-\Psi(t,hz)}{h} \leq \eta |z|_{V'} - \varepsilon \tag{A.21}
$$

so that

$$
\sup_{t\geq 0}\langle \Psi_x(t,0), -z\rangle < +\infty,
$$

and the proof is complete. ■

A.2. Proof of Theorem 5.6

We divide the long proof into several steps. Let $x \in V'$ be fixed, $|x|_{V'} \leq r$, and let $0 \leq t_1 < t_2$.

Claim 1: Let $\varepsilon > 0$ be fixed and let $u_\varepsilon \in L^p_\lambda(0,t;U)$ be ε-optimal at starting point $(0,x)$ with horizon t, and $y_\varepsilon(s) := y(s;0,x,u_\varepsilon)$ be the associated trajectory. Then there exists a bounded continuous function ρ, depending only from r, with $\lim_{t\to+\infty}\rho(t) = 0$, and such that:

$e^{-\lambda t}|y_\varepsilon(t)|^2_{V'} \le \rho(t)$, *if Assumptions 5.1 [8.a] are satisfied;*
$e^{-\lambda t}|y_\varepsilon(t)|_{V'} \le \rho(t)$, *if Assumptions 5.1 [8.b] are satisfied.*
Indeed, applying (A.7) and Lemma A.7, we derive

$$e^{-\lambda t}|y_\varepsilon(t)|_{V'} \le K_r e^{-(\lambda-\omega)t}\left[1+\theta(0,t)^{\frac{1}{q}}\right], \tag{A.22}$$

with K_r a suitable constant. Hence for a (possibly different) constant $K_r > 0$, we have

$$e^{-\lambda t}|y_\varepsilon(t)|^2_{V'} \le K_r e^{-(\lambda-2\omega)t}(1+\theta(0,t)^{\frac{1}{q}}+\theta(0,t)^{\frac{2}{q}}). \tag{A.23}$$

By proceeding as in the proof of Lemma A.5 one sees that the following functions are infinitesimal as t goes to $+\infty$

$$e^{-(\lambda-\omega)t}\theta(0,t)^{\frac{1}{q}}, \quad e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{1}{q}},$$

so that what is left to show is

$$e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{2}{q}}, \quad t \to 0.$$

The property is straightforward in the case $\lambda = \omega p$, as

$$e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{2}{q}} = e^{-(\lambda-2\omega)t}|t|^{\frac{2}{q}},$$

while for $\lambda < \omega p$ one has

$$e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{2}{q}} \le Ce^{-(\lambda-2\omega)t}.$$

Finally, if $\lambda > \omega p$,

$$e^{-(\lambda-2\omega)t}\theta(0,t)^{\frac{2}{q}} \le Ce^{-(\lambda-2\omega)t}e^{2\left(\frac{\lambda}{p}-\omega\right)t} = Ce^{\frac{\lambda}{p}(2-p)t}$$

and Claim 1 is proved.
Claim 2: $\lim_{t_1\to+\infty, t_1<t_2} \Psi(t_1,x) - \Psi(t_2,x) \le 0$
Let $\varepsilon > 0$ be arbitrarily fixed, let $u_\varepsilon \in L^p_\lambda(0,t_2;U)$ be such that

$$\Psi(t_2,x) \ge J_{t_2}(0,x,u_\varepsilon) - \epsilon$$

and let $y_\varepsilon(\tau) := y(\tau;0,x,u_\varepsilon)$. Then

$$\begin{aligned} \Psi(t_2,x) &\ge \int_0^{t_1} e^{-\lambda s}[g_0(y_\varepsilon(s)) + h_0(u_\varepsilon(s))]ds + e^{-\lambda t_1}J_{t_2-t_1}(0,y_\varepsilon(t_1),u_\varepsilon(\cdot+t_1)) - \varepsilon \\ &\ge \Psi(t_1,x) - e^{-\lambda t_1}\phi_0(y_\varepsilon(t_1)) + e^{-\lambda t_1}\Psi(t_2-t_1,y_\varepsilon(t_1)) - \varepsilon \end{aligned} \tag{A.24}$$

where we used (A.6). Consequently, when [8.a] holds

$$\Psi(t_1,x) - \Psi(t_2,x) \le Ce^{-\lambda t_1}(1+|y_\varepsilon(t_1)|^2_{V'}) + \varepsilon \tag{A.25}$$

while for [8.b]

$$\Psi(t_1,x) - \Psi(t_2,x) \le Ce^{-\lambda t_1}(1+|y_\varepsilon(t_1)|_{V'}) + \varepsilon \tag{A.26}$$

for a suitable constant C, and the last implies Claim 2 as a consequence of Claim 1, and Lemma A.8.

Claim 3: $\lim_{t_1 \to +\infty, t_1 < t_2} \Psi(t_2, x) - \Psi(t_1, x) \leq 0$

If we choose $\varepsilon > 0$ and $v_\varepsilon \in L^p_\lambda(0, t_1; U)$ so that

$$\Psi(t_1, x) \geq J_{t_1}(0, x, v_\varepsilon) - \varepsilon,$$

and set $y_\varepsilon(s) := y(s, 0, x, u_\varepsilon)$, then by the DDP contained in (A.5) we obtain

$$\Psi(t_2, x) - \Psi(t_1, x) \leq e^{-\lambda t_1} \Psi(t_2 - t_1, y_\varepsilon(t_1)) - e^{-\lambda t_1} \phi_0(y_\varepsilon(t_1)) + \varepsilon \tag{A.27}$$

which leads as before to the conclusion. Since the constants involved in the estimates are uniform in x, for x varying in a bounded subset of V', we derive the convergence $\Psi(t, x) \to \Psi_\infty(x)$, as $t \to +\infty$ is uniform on bounded subsets of V'.

Next we discuss the convergence of gradients.

Claim 4: Ψ_∞ is Frechét differentiable, with differential Ψ'_∞ and, for every fixed $x \in V'$

$$\Psi_x(t, x) \to \Psi'_\infty(x), \text{ weakly in } V, \text{ as } t \to +\infty.$$

Let x be fixed in V', h a real parameter, with $h \in [-1, 1]$, y in V' with $|y|_{V'} \leq 1$, and $\xi_t(h; x, y) \equiv \xi_t(h) := \Psi(t, x + hy)$. Then

$$\xi'_t(h) := \langle \Psi_x(t, x + hy), y \rangle.$$

Note that, since $\xi_t(h) \to \xi_\infty(h) \equiv \Psi_\infty(x + hy)$ as $t \to +\infty$, if we show that $\xi'_t(h)$ converges uniformly in $[-1, 1]$ to some function as $t \to +\infty$ (or along a subsequence), then such function is $\xi'_\infty(h)$. We do so by means of Ascoli–Arzelà Theorem. We have

$$|\xi'_t(h) - \xi'_t(k)| \leq |y|^2_{V'} [\Psi_x(t))]_L |h - k|_{\mathbb{R}}$$

which implies, by Lemma A.9 (i), that the family $\{\xi'_t\}_{t \geq 0}$ is equicontinuous (more precisely, equilipschitzean). Moreover

$$\begin{aligned} |\xi'_t(h)| &\leq |y|_{V'} |\Psi_x(t, x + hy)|_V \\ &\leq |y|_{V'} \left([\Psi_x(t)]_L |x + hy|_{V'} + |\Psi_x(t, 0)|_V \right) \end{aligned} \tag{A.28}$$

from which follows, by means of Lemma A.9 (ii), that $\{\xi'_t\}_{t \geq 0}$ is uniformly bounded. Consequently, Ψ_∞ is Gateaux differentiable. Indeed, there exists the following limit

$$\lim_{h \to 0} \frac{\Psi_\infty(x + hy) - \Psi_\infty(x)}{h} = \xi'_\infty(0) =: \langle \nabla \Psi_\infty(x), y \rangle_{V'},$$

where $\nabla \Psi_\infty$ indicates the Gateaux differential of Ψ_∞. In particular, what we prove implies also

$$\Psi_x(t, x) \to \nabla \Psi_\infty(x) \text{ weakly in } V \text{ as } t \to +\infty.$$

Finally we show that $\nabla \Psi_\infty$ is continuous. It suffices to pass to limits as t goes to ∞ in

$$\begin{aligned} |\xi'_t(0; x, y) - \xi'_t(0; z, y)| &\leq |y|_{V'} |\Psi_x(t, x) - \Psi(t, z)|_V \\ &\leq |y|_{V'} \sup_{t \geq 0} [\Psi_x(t)]_L |x - z|_{V'}. \end{aligned} \tag{A.29}$$

Hence Ψ_∞ is Frechét differentiable with Frechét differential $\Psi'_\infty = \nabla \Psi_\infty$. The proof that Ψ_∞ is convex and in $C^1_{Lip}(V')$ is trivial by means of Lemma A.9

A.3. Proof of Theorem 5.7 (i)

First of all we show that, for any fixed t, x and $u \in L^p_\lambda(0,+\infty)$ we have

(A.30)
$$\exists \lim_{t\to+\infty} J_t(0,x,u) = J_\infty(0,x,u).$$

We separately show that, if $y(s) = y(s;0,x,u)$, then
(A.31)
$$\lim_{t\to+\infty} \left|e^{-\lambda t}\phi_0(y(t))\right| = 0, \quad \text{and} \quad J_\infty(t,x,u) = \lim_{t\to+\infty} \int_0^t e^{-\lambda\tau}[g_0(y(\tau)) + h_0(u(\tau))]d\tau.$$

Indeed
$$|y(t)|_{V'} \le Ke^{\omega t}(1+\theta(0,t)^{\frac{1}{q}})$$
where $K = |x|_{V'} \vee \|u\|_{L^p_\lambda(0,+\infty;U)}$. Hence

(A.32)
$$e^{-\lambda t}|y(t)|^2_{V'} \le \rho(t), \quad e^{-\lambda t}|y(t)|_{V'} \le \rho(t)$$

where $\rho(t)$ denotes some positive function with $\lim_{t\to+\infty} \rho(t) = 0$ (obtained as in proof of Claim 1 of Theorem 5.6). These inequalities combined with Assumptions 5.1 [8.a] or [8.b] – recall that ϕ_0 and g_0 are either sublinear or subquadratic – give the first equality in (A.31), and by means of dominated convergence also
$$\int_0^t e^{-\lambda s} g_0(y(s))ds \to \int_0^{+\infty} e^{-\lambda s} g_0(y(s))ds, \quad as \quad t \to +\infty.$$
To complete the proof of (A.31) we just observe
$$\int_0^t e^{-\lambda s} h_0(u(s))ds \to \int_0^{+\infty} e^{-\lambda s} h_0(u(s))ds, \quad as \quad t \to +\infty$$
by monotone convergence, for h_0 is bounded from below.

Then (A.30) is proved. As a consequence, by passing to limits as t tends to $+\infty$ in
$$\Psi(t,x) \le J_t(0,x,u), \quad \forall u \in L^p_\lambda(0,+\infty;U)$$
we obtain
$$\Psi_\infty(x) \le J_\infty(0,x,u), \quad \forall u \in L^p_\lambda(0,+\infty;U)$$
which implies
$$\Psi_\infty(x) \le Z_\infty(0,x).$$
We now need to show that the reverse inequality holds. Let $\varepsilon > 0$ be fixed. From (A.15) and (A.18) we derive that if u^* is ε-optimal at $(0,x)$ with horizon $+\infty$

(A.33)
$$J_\infty(0,x,u^*) \le C(1+|x|^2_{V'}) + \varepsilon$$

with C a suitable constant. Hence, we set

(A.34)
$$u_t(s) = \begin{cases} u_1(s) & s \in [0,t] \\ u_2(s) & s \in]t,+\infty[\end{cases}$$

where $u_1 \in L^p_\lambda(0,t;U)$ is ε-optimal for J_t at $(0,x)$, and $u_2 \in L^p_\lambda(t,+\infty;U)$ is ε-optimal for J_∞ at $(0, y_t(t))$, with $y_t(s) := y(s; 0, x, u_t)$, and we derive by means of (A.6) and (A.33) the following chain of inequalities

$$
\begin{aligned}
Z_\infty(0,x) &\le J_\infty(0,x,u_t) \\
&= J_t(0,x,u_1) + e^{-\lambda t}\big[J_\infty(0,y_t(t),u_2) - \phi_0(y_t(t))\big] \\
&\le \Psi(t,x) + Ce^{-\lambda t}(1 + |y_t(t)|_{V'} + |y_t(t)|^2_{V'}) + 2\varepsilon \\
&=: \Psi(t,x) + \rho(t) + 2\varepsilon
\end{aligned}
\tag{A.35}
$$

and with C some suitable constant (possibly different from the one mentioned above). Note that $\rho(t) \to 0$, as $t \to +\infty$ as one derives from Claim 1 in the proof of Theorem 5.6. Hence, by passing to limits as t goes to $+\infty$, we derive

$$
Z_\infty(0,x) \le \Psi_\infty(x),
$$

and the proof is complete.

A.4. Proof of Theorem 5.7 *(ii)*

To prove the theorem we make use of the Dynamic Programming Principle (DPP from now on) contained in the following Lemma.

Lemma A.10. *We have*

$$
\Psi_\infty(x) = \inf_{u \in L^p_\lambda(0,+\infty;U)} \left\{ \int_0^t e^{-\lambda s}(g_0(y(s)) + h_0(u(s)))ds + e^{-\lambda t}\Psi_\infty(y(t)) \right\}, \quad \forall t > 0.
$$

Moreover, given any $\varepsilon > 0$, if u_ε is such that

$$
J_\infty(0,x,u_\varepsilon) < \Psi_\infty(x) + \varepsilon
$$

then also

$$
\int_0^t e^{-\lambda s}(g_0(y(s)) + h_0(u(s)))ds + e^{-\lambda t}\Psi_\infty(y(t)) < \Psi_\infty(x) + \varepsilon
$$

The proof of this lemma is standard and we omit it.

We prove then Theorem 5.7 *(ii)*. Let $t > 0$ be fixed. Let also $u(s) \equiv \bar{u} \in dom(h_0)$, and let $\bar{y}(s) := y(s; 0, x, \bar{u})$. Then the DPP implies

$$
\frac{e^{-\lambda t}\Psi_\infty(\bar{y}(t)) - \Psi_\infty(x)}{t} \ge -\frac{1}{t}\int_0^t e^{-\lambda s}[g_0(\bar{y}(s)) + h_0(\bar{u})]ds.
\tag{A.36}
$$

Since

$$
\frac{\bar{y}(t) - \bar{y}(0)}{t} = \frac{e^{At}x - x}{t} + \frac{1}{t}\int_0^t e^{A(t-s)}B\bar{u}ds \to Ax + B\bar{u}, \quad \textit{as} \quad t \to 0,
$$

and $s \mapsto g_0(y(s))e^{-\lambda s} \in C(0,t;U)$, then we may pass to limits in (A.36) and obtain

$$
-\lambda\Psi_\infty(x) + \langle \Psi'_\infty(x), Ax\rangle + \langle \Psi'_\infty(x), B\bar{u}\rangle + h_0(\bar{u}) + g_0(x) \ge 0,
$$

and take the infimum of both sides as $\bar{u} \in dom(h_0)$ and derive

$$-\lambda\Psi_\infty(x) + \langle \Psi'_\infty(x), Ax\rangle - h_0^*(-B^*\Psi'_\infty(x)) + g(x) \geq 0.$$

Next we prove the reverse inequality. Let $\varepsilon > 0$ and $t \in [0,1]$ arbitrarily fixed, and let u_ε be an εt- optimal control at $(0, x)$ with horizon $+\infty$, and y_ε be the associated trajectory. Then by the DPP

$$e^{-\lambda t}\Psi_\infty(y_\varepsilon(t)) - \Psi_\infty(x) + \int_0^t e^{-\lambda s}[g_0(y_\varepsilon(s)) + h_0(u_\varepsilon(s))]ds \leq \varepsilon t, \ \forall t \in [0,1].$$

Since Ψ_∞ is convex and differentiable, the preceding implies

$$\begin{aligned} e^{-\lambda t}\langle \Psi'_\infty(x), \frac{y_\varepsilon(t) - x}{t}\rangle - \Psi_\infty(x)\frac{1 - e^{-\lambda t}}{t} + & \\ + \frac{1}{t}\int_0^t e^{-\lambda s}[g_0(y_\varepsilon(s)) + h_0(u_\varepsilon(s))]ds \leq \varepsilon, \ \forall t \in [0,1]. \end{aligned} \tag{A.37}$$

Now we show that

$$\frac{1}{t}\int_0^t e^{-\lambda s} g_0(y_\varepsilon(s))ds = g_0(x) + \rho(t) \tag{A.38}$$

where by $\rho(t)$, we denote some real function not depending on u_ε such that $\rho(t) \to 0$ as $t \to 0$. Indeed the assumptions on g_0 imply that

$$|g_0(x) - g_0(y)| \leq C(|x|_{V'}, |y|_{V'})|x - y|_{V'}$$

with

$$C(\alpha, \beta) := \big([g_0']_L + |g_0'(0)|\big)\big(1 + \alpha \vee \beta\big).$$

Moreover by Lemma A.7 and by (A.7) we derive

$$|y_\varepsilon(s) - x|_{V'} \leq |e^{sA}x - x|_{V'} + C(1 + |x|^2_{V'})e^{\omega s}\theta(0, s)^{\frac{1}{q}} \tag{A.39}$$

for some constant C (independent of u_ε and x) and with θ the function defined in Lemma A.5, which has as a consequence

$$\sup_{s\in[0,1]} |y_\varepsilon(s)|_{V'} < K(x) < +\infty,$$

with $K(x)$ not depending on u_ε. Hence for $t \in [0,1]$

(A.40)

$$\begin{aligned} \frac{1}{t}\int_0^t \left|e^{-\lambda s} g_0(y_\varepsilon(s))ds - g_0(x)\right| ds &\leq \frac{1}{t}\int_0^t e^{-\lambda s}|g_0(y_\varepsilon(s)) - g_0(x)|ds + \rho(t) \\ &\leq C(|x|_{V'}, K(x))\left[C(1 + |x|^2_{V'})\frac{1}{t}\int_0^t e^{-(\lambda-\omega)s}\theta(0, s)^{\frac{1}{q}}ds + \frac{1}{t}\int_0^t e^{-\lambda s}|e^{sA}x - x|_{V'}ds\right] \\ &\quad + \rho(t), \end{aligned}$$

which implies (A.38) by definition of $\theta(0,s)$.

Observe now that, as $x \in D(A)$, then

$$\frac{y_\varepsilon(t)-x}{t} = Ax + \rho(t) + \frac{1}{t}\int_0^t e^{(t-s)A} B u_\varepsilon(s) ds.$$

Then, the last and (A.38) imply that (A.37) can be written as

$$\begin{aligned}
&\langle \Psi'_\infty(x), Ax\rangle - \lambda \Psi_\infty(x) + g(x) + \\
&+ \frac{1}{t}\int_0^t e^{-\lambda s}\big[\langle e^{-\lambda(t-s)} B^* e^{(t-s)A^*}\Psi'_\infty(x), u_\varepsilon(s)\rangle + h_0(u_\varepsilon(s))\big] ds \le \varepsilon + \rho(t), \\
&\forall t \in [0,1].
\end{aligned} \tag{A.41}$$

We then get that

$$\begin{aligned}
&\frac{1}{t}\int_0^t e^{-\lambda s}\big[\langle e^{-\lambda(t-s)} B^* e^{(t-s)A^*}\Psi'_\infty(x), u_\varepsilon(s)\rangle + h_0(u_\varepsilon(s))\big] ds \ge \\
&\ge -\frac{1}{t}\int_0^t e^{-\lambda s}\sup_{u\in U}\big[\langle -e^{-\lambda(t-s)} B^* e^{(t-s)A^*}\Psi'_\infty(x), u\rangle - h_0(u)\big] ds = \\
&= -\frac{1}{t}\int_0^t e^{-\lambda s} h_0^*\left(-e^{-\lambda(t-s)} B^* e^{(t-s)A^*}\Psi'_\infty(x)\right) ds \\
&= -h_0^*(-B^*\Psi'_\infty(x)) + \rho(t),
\end{aligned} \tag{A.42}$$

for $h_0^* \in C^1_{Lip}(U)$ by assumption. Hence

$$\langle \Psi'_\infty(x), Ax\rangle - \lambda \Psi_\infty(x) + g(x) - h_0^*(-B^*\Psi'_\infty(x)) \le \varepsilon + \rho(t), \quad \forall t \in [0,1],$$

which implies the thesis by passing to limits as $t \to 0$.

A.5. Verification Theorem

We state and prove the following verification theorem:

Theorem A.11. *Let Assumptions 5.1 hold. Let $t \ge 0$ and $x \in V'$ be fixed. Then*

$$e^{-\lambda t}\Psi_\infty(x) = J_\infty(t,x,u) - \int_t^T e^{-\lambda s}\big[h_0^*(-B^*\Psi'_\infty(y(s))) + (B^*\Psi'_\infty(y(s)) \mid u(s))_U + h_0(u(s))\big] ds. \tag{A.43}$$

As a consequence, an admissible pair (u,y) at (t,x) is optimal if and only if

$$\sup_{u\in U}\{(u| - B^*\Psi'_\infty(y(s)))_U - h_0(u)\} = (u(s)| - B^*\Psi'_\infty(y(s)))_U - h_0(u(s))$$

for a.e. $s \ge 0$, which is equivalent to

$$u(s) = (h_0^*)'(-B^*\Psi'_\infty(y(s))$$

for a.e. $s \ge 0$.

Proof. Let first $x \in D(A)$, $t > 0$ be fixed. Let u be any admissible control at (t, x) such that $J_T(t, x, u) < +\infty$ for every $T > 0$. (Note that an admissible control violating this condition cannot be optimal for the infinite horizon problem, as $J_T(t, x, u) = +\infty$ for some $T > 0$ implies $J_\infty(t, x, u) = +\infty$.) Let y be the associated trajectory. Then for a.e. $s \in [t, +\infty[$ we may differentiate $e^{-\lambda s}\Psi_\infty(y(s))$ as function of s to obtain

(A.44)
$$\begin{aligned}&\frac{d}{ds}e^{-\lambda s}\Psi_\infty(y(s)) = -\lambda e^{-\lambda s}\Psi_\infty(y(s)) + e^{-\lambda s}\langle \Psi'_\infty(y(s)), Ay(s) + Bu(s)\rangle \\ &= h_0^*(-B^*\Psi'_\infty(y(s))) - g_0(y(s)) + (B^*\Psi'_\infty(y(s)) \mid u(s))_U + e^{-\lambda s}h_0(u(s)) - e^{-\lambda s}h_0(u(s))\end{aligned}$$

where we used the fact that Ψ_∞ solves the stationary HJB equation, and we added and subtracted the term $e^{-\lambda s}h_0(u(s))$. Integrating such equation on $[t, T]$ we have

$$\begin{aligned}&e^{-\lambda T}\Psi_\infty(y(T)) - e^{-\lambda t}\Psi_\infty(x) = \\ &= \int_t^T e^{-\lambda s}\big[h_0^*(-B^*\Psi'_\infty(y(s))) + (B^*\Psi'_\infty(y(s)) \mid u(s))_U + h_0(u(s))\big]\big]ds+ \\ &\qquad - J_T(t, x, u) + e^{-\lambda T}\phi_0(y(T)).\end{aligned} \tag{A.45}$$

Such relation holds for all admissible controls. If now we show that, for any fixed admissible control u, is

$$e^{-\lambda T}\Psi_\infty(y(T)) \to 0, \ \textit{and}\ e^{-\lambda T}\phi_0(y(T)) \to 0, \ \textit{as}\ T \to +\infty,$$

then we may pass to limits in (A.45) and derive (A.43). Indeed, using the fact that ϕ_0 and Ψ_∞ are either sublinear or subquadratic, we observe that, by (A.32)

$$e^{-\lambda T}|\Psi_\infty(y(T)) + \phi_0(y(T))| \le CK\rho(T),$$

where K and ρ are the same there considered, and C a suitable positive constant. Then $(A.43)$ is proved. It also implies that $\Psi_\infty(x) \le J_\infty(0, x, u)$ and that the equality holds if and only if

$$\int_t^{+\infty} e^{-\lambda s}\big[h_0^*(-B^*\Psi'_\infty(y(s))) + (B^*\Psi'_\infty(y(s)) \mid u(s))_U + h_0(u(s))\big]\big]ds = 0$$

which means, by the positivity of the integrand that

$$h_0^*(-B^*\Psi'_\infty(y(s))) = (-B^*\Psi'_\infty(y(s)) \mid u(s))_U - h_0(u(s))$$

for almost every $s \ge t$. The claim easily follows from the definition of h_0^*. The claim for generic $x \in V'$ follows by approximating it by a sequence of elements of $D(A)$ and observing that the relation (A.43) make sense also for $x \in V'$. ∎

A.6. Proof of Theorem 5.8

We first observe that the closed loop equation has a unique solution y^* since the feedback map defined as

$$G(x) = (h_0^*)'(-B^*\Psi_\infty(x))$$

is Lipschitz continuous, as one may show by a standard fixed point argument.

Next we prove that the control

$$u^*(s) = (h_0^*)'(-B^*\Psi'_\infty(y^*(s))$$

is admissible, i.e. it belongs to $L^p_\lambda(t,+\infty;U)$. To do so, we first observe that the relation (A.45) holds true for any control $u \in L^1_{loc}(t,+\infty;U)$ such that $J_T(t,x,u) < +\infty$ for every $T > 0$. Since by definition we have

$$h_0(u^*(s)) = (-B^*\Psi'_\infty(y^*(s))|u^*(s))_U - h_0^*\left(-B^*\Psi'_\infty(y^*(s))\right)$$

then also $J_T(t,x,u^*) < +\infty$ for every $T > 0$. So we get that u^* satisfies

$$J_T(t,x,u^*) - e^{-\lambda T}\phi_0(y^*(T)) = e^{-\lambda t}\Psi_\infty(x) - e^{-\lambda T}\Psi_\infty(y^*(T)). \tag{A.46}$$

By means of (A.8) and proceeding as in the proof of Lemma A.5, one derives

$$\int_t^T e^{-\lambda s}|y^*(s)|_{V'} ds \le \gamma_1 + \gamma_2\|u^*\|_{L^p_\lambda(t,T;U)}$$

where γ_1, γ_2 are suitable constants (depending on x, t), so that

$$\begin{aligned} A :&= J_T(t,x,u^*) - e^{-\lambda T}\phi_0(y^*(T)) \\ &\ge \int_t^T e^{-\lambda s}[a|u^*(s)|_U + b]ds - C\int_t^T e^{-\lambda s}[1 + |y^*(s)|_{V'}]ds \\ &\ge a\|u^*\|^p_{L^p_\lambda(t,T;U)} + \frac{b-C}{\lambda} - \gamma_1 - \gamma_2\|u^*\|_{L^p_\lambda(t,T;U)} \\ &\ge a\|u^*\|^p_{L^p_\lambda(t,T;U)} - \gamma_2\|u^*\|_{L^p_\lambda(t,T;U)} - \gamma_3 \end{aligned} \tag{A.47}$$

for a suitable constant γ_3. On the other hand, since by means of (A.8), and proceeding again like in the proof of Lemma A.5, we have

$$e^{-\lambda T}(1 + |y^*(T)|_{V'}) \le \gamma_4 + \gamma_5\|u^*\|_{L^p_\lambda(t,T;U)}$$

where γ_4, γ_5 are suitable constants (depending on x, t), so that, using Lemma A.8

$$\begin{aligned} B :=& e^{-\lambda t}\Psi_\infty(x) - e^{-\lambda T}\Psi_\infty(y^*(T)) \\ &\le e^{-\lambda t}\Psi_\infty(x) - C(\gamma_4 + \gamma_5\|u^*\|_{L^p_\lambda(t,T;U)}) \end{aligned} \tag{A.48}$$

Hence combining this inequality with (A.47) and (A.48) we obtain

$$a\|u^*\|^p_{L^p_\lambda(t,T;U)} - \gamma\|u^*\|_{L^p_\lambda(t,T;U)} \le \eta$$

for suitable positive constants γ and η independent of T. Then we may pass to limits as T goes to $+\infty$ and derive

$$a\|u^*\|^p_{L^p_\lambda(t,+\infty;U)} - \gamma\|u^*\|_{L^p_\lambda(t,+\infty;U)} \le \eta,$$

which implies that $u^* \in L^p_\lambda(t,+\infty;U)$, and is hence admissible.

Then by the above Theorem A.11 we get that the couple (u^*, y^*) is optimal. The uniqueness follows by the uniqueness of the solution of the closed loop equation.

A.7. Proof of Thorem 5.7 *(iii)*

Let $\psi \in C^1$ be any other function that satisfies the stationary HJB equation (5.4) in classical sense as from Definition 5.3. Then we have, arguing exactly as in the proof of Theorem A.11, that ψ satisfies the fundamental relation (A.43) in place of Ψ_∞. This implies, setting $t = 0$, that $\psi(x) \le J_\infty(0, x, u)$ and consequently

$$\psi(x) \le \Psi_\infty(x).$$

Moreover if we find an admissible control u at x such that

$$(h_0^*)'(-B^*\psi'(y(s))) = (-B^*\psi'(y(s)) \mid u(s))_U - h_0(u(s))$$

then $\psi(x) = J_\infty(0, x, u)$, u is optimal, and then $\psi(x) = \Psi_\infty(x)$. Such a control exists as one may derive arguing as in the proof of the Theorem 5.8.

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